

Invertible harmonic mappings of unit disk onto Dini smooth Jordan domains

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Abstract

In this paper we extend Rado-Choquet-Kneser theorem for the mappings with weak homeomorphic Lipschitz boundary data and Dini's smooth boundary but without restriction on the convexity of image domain, provided that the Jacobian satisfies a certain boundary condition. The proof is based on a recent extension of Rado-Choquet-Kneser theorem by Alessandrini and Nesi [1] and it is used the approximation principle.

1 Introduction

Harmonic mappings in the plane are univalent complex-valued harmonic functions of a complex variable. Conformal mappings are a special case where the real and imaginary parts are conjugate harmonic functions, satisfying the Cauchy-Riemann equations. Harmonic mappings were studied classically by differential geometers because they provide isothermal (or conformal) coordinates for minimal surfaces. More recently they have been actively investigated by complex analysts as generalizations of univalent analytic functions, or conformal mappings. For the background to this theory we refer to the book of Duren [4]. If w is a univalent complex-valued harmonic functions, then by Lewy's theorem (see [11]), w has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, w is a diffeomorphism. Moreover, if w is a harmonic mapping of the unit disk \mathbf{U} onto a convex Jordan domain Ω , mapping the boundary $\mathbf{T} = \partial\mathbf{U}$ onto $\partial\Omega$ homeomorphically, then w is a diffeomorphism. This is celebrated theorem of Rado, Kneser and Choquet ([10]). This theorem has been extended in various

directions (see for example [8], [2], [12] and [13]). One of the recent extensions is the following proposition, due to Nesi and Alessandrini [1], which is one of the main tools in proving our main result.

Proposition 1.1. *Let $F : \mathbf{T} \rightarrow \gamma \subset \mathbf{C}$ be an orientation preserving diffeomorphism of class C^1 onto a simple closed curve. Let D be a bounded domain such that $\partial D = \gamma$. Let $w = P[F] \in C^1(\bar{\mathbf{U}}; \mathbf{C})$. The mapping w is a diffeomorphism of \mathbf{U} onto D if and only if*

$$J_w(\zeta) > 0 \text{ everywhere on } \mathbf{T}, \quad (1)$$

where $J_w(\zeta) := \lim_{r \rightarrow 1^-} J_w(r\zeta)$, and $J_w(r\zeta)$ is the Jacobian of w at $r\zeta$.

2 Statement of the main result and preliminaries

In this paper we generalize Rado-Kneser-Choquet theorem as follows.

Theorem 1 (The main result). *Let $F : \mathbf{T} \rightarrow \gamma \subset \mathbf{C}$ be an orientation preserving Lipschitz weak homeomorphism of the unit circle \mathbf{T} onto a Dini's smooth Jordan curve and let $w = P[F]$ be its Poisson extension in the unit disk. Let D be a bounded domain such that $\partial D = \gamma$. Then there is a continuous function $T(\zeta)$ on \mathbf{T} such that*

$$J_w(\zeta) = |\partial_t F(\zeta)| T(\zeta), \quad \text{for a.e. } \zeta = e^{it} \in \mathbf{T}.$$

If

$$T(\zeta) > 0 \quad \text{in } \mathbf{T}, \quad (2)$$

then the mapping $w = P[F]$ is a diffeomorphism of \mathbf{U} onto D .

In order to compare this statement with Kneser's Theorem, it is worth noticing that, when D is convex, then by Remark 3.1 the condition (2) is automatically satisfied. In a paper of the author in [9] the author proved Theorem 1 under a stronger condition that $\gamma \in C^{1,\alpha}$ for some $\alpha > 0$ and by using a different approach.

Note that we do not have any restriction on convexity of image domain in Theorem 1 which is proved in section 3.

A conjecture

Motivated by Theorem 1 we state the following conjecture. Let $F : \mathbf{T} \rightarrow \gamma \subset \mathbf{C}$ be a homeomorphism, where γ is a rectifiable Jordan curve. Let D be the bounded

domain such that $\partial D = \gamma$. The mapping $w = P[F]$ is a diffeomorphism of \mathbf{U} onto D if and only if

$$\text{ess inf}\{J_w(\zeta) : \zeta \in \mathbf{T}\} \geq 0. \quad (3)$$

Before proving results we overview the involved concepts and make the main definitions concerning this paper.

2.1 Smooth Jordan curves and their parameterizations

Suppose that γ is a rectifiable curve in the complex plane. Denote by l the length of γ and let $g : [0, l] \mapsto \gamma$ be the arc length parameterization of γ , i.e. the parameterization satisfying the condition: $|g'(s)| = 1$ for all $s \in [0, l]$.

Let

$$K(s, t) = \text{Re} [\overline{(g(t) - g(s))} \cdot ig'(s)] \quad (4)$$

be a function defined on $[0, l] \times [0, l]$. By $K(s \pm l, t \pm l) = K(s, t)$ we extend it on $\mathbf{R} \times \mathbf{R}$. Note that $ig'(s)$ is the inner unit normal vector of γ at $g(s)$ and therefore, if γ is convex then

$$K(s, t) \geq 0 \text{ for every } s \text{ and } t. \quad (5)$$

Suppose now that $F : \mathbf{R} \mapsto \gamma$ is an arbitrary 2π periodic Lipschitz function such that $F|_{[0, 2\pi)} : [0, 2\pi) \mapsto \gamma$ is an orientation preserving bijective function. Then there exists an increasing continuous function $f : [0, 2\pi] \mapsto [0, l]$ such that

$$F(\tau) = g(f(\tau)). \quad (6)$$

In the remainder of this paper we will identify $[0, 2\pi)$ with the unit circle \mathbf{T} , and $F(s)$ with $F(e^{is})$. In view of the previous convention we have

$$F'(\tau) = g'(f(\tau)) \cdot f'(\tau),$$

and therefore

$$|F'(\tau)| = |g'(f(\tau))| \cdot |f'(\tau)| = f'(\tau).$$

Along with the function K we will also consider the function K_F defined by

$$K_F(t, \tau) = \text{Re} [\overline{(F(t) - F(\tau))} \cdot iF'(\tau)].$$

It is easy to see that

$$K_F(t, \tau) = f'(\tau)K(f(t), f(\tau)). \quad (7)$$

Definition 2.1. Let $l = |\gamma|$. We will say that a surjective function $F = g \circ f : \mathbf{T} \rightarrow \gamma$ is a weak homeomorphism, if $f : [0, 2\pi] \rightarrow [0, l]$ is a nondecreasing surjective function.

Definition 2.2. Let $f : [a, b] \rightarrow \mathbf{C}$ be a continuous function. The modulus of continuity of f is

$$\omega(t) = \omega_f(t) = \sup_{|x-y| \leq t} |f(x) - f(y)|.$$

The function f is called Dini-continuous if

$$\int_{0^+} \frac{\omega_f(t)}{t} dt < \infty. \quad (8)$$

Here $\int_{0^+} := \int_0^k$ for some positive constant k . A C^1 Jordan curve γ with the length $l = |\gamma|$, is said to be Dini smooth if g' is Dini continuous. We say that γ is of class $C^{1,\mu}$, $0 < \mu \leq 1$, if g is of class C^1 and

$$\sup_{t \neq s} \frac{|g'(t) - g'(s)|}{|t - s|^\mu} < \infty.$$

Observe that every smooth $C^{1,\mu}$ Jordan curve is Dini smooth.

Lemma 1. [9] If γ is Dini smooth, and ω is modulus of continuity of g' , then

$$|K(s, t)| \leq \int_0^{\min\{|s-t|, l-|s-t|\}} \omega(\tau) d\tau. \quad (9)$$

A function $\varphi : A \rightarrow B$ is called (ℓ, \mathcal{L}) bi-Lipschitz, where $0 < \ell < \mathcal{L} < \infty$, if $\ell|x - y| \leq |\varphi(x) - \varphi(y)| \leq \mathcal{L}|x - y|$ for $x, y \in A$.

2.2 Harmonic mappings

A mapping w is called *harmonic* in a region D if $w = u + iv$ where u and v are harmonic functions in D . If D is simply-connected, then there are two analytic functions g and h defined on D such that

$$w = g + \bar{h}.$$

Let

$$P(r, t) = \frac{1 - r^2}{2\pi(1 - 2r \cos t + r^2)}$$

denote the Poisson kernel. The Poisson integral of $F \in L^1(\mathbf{T})$ is a harmonic function given by

$$w(z) = P[F](z) = \int_0^{2\pi} P(r, t - \tau) F(e^{it}) dt, \quad (10)$$

where $z = re^{i\tau} \in \mathbf{U}$.

3 The proof of the main theorem

The aim of this chapter is to prove Theorem 1. As in [9], we will construct a suitable sequence w_n of univalent harmonic mappings, converging almost uniformly to $w = P[F]$. In order to do so, we will mollify the boundary function F , by a sequence of diffeomorphism F_n and take the Poisson extension $w_n = P[F_n]$. We will show that, under the condition of Theorem 1 for large n , w_n satisfies the conditions of theorem of Alessandrini and Nesi. By a result of Hengartner and Schober [6], the limit function w of a locally uniformly convergent sequence of univalent harmonic mappings w_n is univalent, providing that F is a surjective mapping.

We begin by the following lemmata.

Lemma 2. (See e.g. [9]). *If $\varphi : \mathbf{R} \rightarrow \mathbf{R}$ is a L - Lipschitz weak homeomorphism, such that $\varphi(x + a) = \varphi(x) + b$ for some a and b and every x , then there exist a sequence of L - Lipschitz diffeomorphisms $\varphi_n : \mathbf{R} \rightarrow \mathbf{R}$ such that φ_n converges uniformly to φ , and $\varphi_n(x + a) = \varphi_n(x) + b$.*

Lemma 3. [9] *If $\omega : [0, l] \rightarrow \mathbf{R}_+$ is a bounded function satisfying (8), then for every constant a , $\omega(ax)$ satisfies (8). Next for every $0 < y \leq l$ there hold the following formula:*

$$\int_{0+}^y \frac{1}{x^2} \int_0^x \omega(at) dt dx = \int_{0+}^y \left[\frac{\omega(ax)}{x} - \frac{\omega(ax)}{y} \right] dx. \quad (11)$$

Lemma 4. [9] *If $w = P[F]$ is a harmonic mapping, such that $F = g \circ f$, where g and f are as mentioned earlier, is a Lipschitz weak homeomorphism from the unit circle onto a Dini smooth Jordan curve, then for almost every $\tau \in [0, 2\pi]$*

$$J_w(e^{i\tau}) := \lim_{r \rightarrow 1} J_w(re^{i\tau})$$

and there hold the formula

$$J_w(e^{i\tau}) = f'(\tau) \int_0^{2\pi} \frac{\operatorname{Re} [\overline{(g(f(t)) - g(f(\tau)))} \cdot ig'(f(\tau))]}{2 \sin^2 \frac{t-\tau}{2}} dt. \quad (12)$$

For a Lipschitz non-decreasing function f and an arc-length parametrization g of the Dini's smooth curve γ we define the operator T as follows

$$T[f](\tau) = \int_0^{2\pi} \frac{\operatorname{Re} [\overline{(g(f(t)) - g(f(\tau)))} \cdot ig'(f(\tau))]}{2 \sin^2 \frac{t-\tau}{2}} dt, \tau \in [0, 2\pi]. \quad (13)$$

According to Lemma 4, this integral converges.

Remark 3.1. Notice that, if γ is a convex Jordan curve then $\operatorname{Re} [\overline{(g(f(t)) - g(f(\tau)))} \cdot ig'(f(\tau))] \geq 0$, and therefore $T[f] > 0$. In the next proof, we will show that $T[f]$ is continuous if f is a Lipschitz weak homeomorphism and under the integral condition $T[f] > 0$ the harmonic extension of a bi-Lipschitz mapping is a diffeomorphism regardless of the condition of convexity.

Proof of Theorem 1. Assume for simplicity that $|\gamma| = 2\pi$. The general case follows by normalization. Let $g : [0, 2\pi] \rightarrow \gamma$ be an arc length parametrization of γ . Then $F(e^{it}) = g(f(t))$, where $f : \mathbf{R} \rightarrow \mathbf{R}$ is a L -Lipschitz weak homeomorphism such that $f(t + 2\pi) = f(t) + 2\pi$. By using Lemma 2, we can choose a family of L -Lipschitz diffeomorphisms f_n converging uniformly to f . Then

$$T[f_n] = \int_0^{2\pi} \frac{\operatorname{Re} [\overline{(g(f_n(t)) - g(f_n(\tau)))} \cdot ig'(f_n(\tau))]}{2 \sin^2 \frac{t-\tau}{2}} dt, \tau \in [0, 2\pi]. \quad (14)$$

We are going to show that $T[f_n]$ converges uniformly to $T[f]$. In order to do this we apply Arzela-Ascoli theorem. Let $\beta : [0, 2\pi] \rightarrow \mathbf{R}$ be a function such that $g'(s) = e^{i\beta(s)}$.

As in [9] we obtain

$$T[f](\tau) = \int_{-\pi}^{\pi} f'(t + \tau) \cdot \sin[\beta(f(t + \tau)) - \beta(f(\tau))] \cot \frac{t}{2} \frac{dt}{2\pi}.$$

Then

$$\begin{aligned} |T[f_n]| &\leq \frac{1}{\pi} \|f'_n\| \int_0^{\pi} \omega_{\beta}(\|f'_n\|t) \cot \frac{t}{2} dt \\ &\leq \frac{1}{\pi} \|f'\| \int_0^{\pi} \omega_{\beta}(\|f'\|t) \cot \frac{t}{2} dt = C(f, \gamma) < \infty. \end{aligned}$$

We prove now that $T[f_n]$ is equicontinuous family of functions. In the contrary, there is $x_o \in [0, 2\pi]$ and $\epsilon_o > 0$ and there is a sequence of non-decreasing natural numbers n_k and a sequence of real numbers τ_k tending to zero such that for every $k \in \mathbf{N}$ we have

$$|T[f_{n_k}](x_o + \tau_k) - T[f_{n_k}](x_o)| \geq \epsilon_o. \quad (15)$$

Assume without loss of generality that $x_o = 0$. Use the notation

$$\mathcal{K}_k(t, \tau) = \operatorname{Re} \left[\overline{(g(f_{n_k}(t)) - g(f_{n_k}(\tau)))} \cdot ig'(f_{n_k}(\tau)) \right].$$

On the other hand by (9) and (7), for

$$\sigma_k = \min\{|f_{n_k}(t + \tau_k) - f_{n_k}(\tau_k)|, l - |f_{n_k}(t + \tau_k) - f_{n_k}(\tau_k)|\}$$

we obtain

$$|\mathcal{K}_k(t + \tau_k, \tau_k)| \leq \|f'_{n_k}\| \int_0^{\sigma_k} \omega(u) du,$$

where ω is the modulus of continuity of g' . Since $\|f'_{n_k}\| \leq \|f'\|$ we have

$$\begin{aligned} \frac{|\mathcal{K}_k(t + \tau_k, \tau_k)|}{2 \sin^2 \frac{t}{2}} &\leq \frac{\|f'\|}{2 \sin^2 \frac{t}{2}} \int_0^{\sigma_k} \omega(u) du \\ &\leq \frac{\sigma_k \|f'\| \pi}{t^2} \int_0^t \omega\left(\frac{\sigma_k}{t} u\right) du \\ &\leq \frac{\pi \|f'\|^2}{4t^2} \int_0^t \omega(\|f'\| u) du := Q(t). \end{aligned} \quad (16)$$

Thus $Q(t)$ is a dominant for the expressions

$$\frac{|\mathcal{K}_k(t + \tau_k, \tau_k)|}{2 \sin^2 \frac{t}{2}}.$$

Having in mind the equation (11), we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} |Q(t)| dt &\leq \frac{2\pi \|f'\|^2}{2} \int_0^{\pi} \frac{1}{t^2} \int_0^t \omega(\|f'\| u) du \\ &= \pi \|f'\|^2 \int_0^{\pi} \left(\frac{\omega(\|f'\| u)}{u} - \frac{\omega(\|f'\| u)}{\pi} \right) du \\ &< \infty. \end{aligned}$$

According to the Lebesgue Dominated Convergence Theorem, taking the limit when $k \rightarrow \infty$ under the integral sign in the integral (14) we obtain that $\lim_{k \rightarrow \infty} T_{f_{n_k}}(\tau_k) =$

$T_{f_{n_0}}(0)$ if n_k is a stationary sequence and $\lim_{k \rightarrow \infty} T_{f_{n_k}}(\tau_k) = T_f(0)$ in the other case. Similarly $\lim_{k \rightarrow \infty} T_{f_{n_k}}(0) = T_{f_{n_0}}(0)$ if n_k is a stationary sequence and $\lim_{k \rightarrow \infty} T_{f_{n_k}}(0) = T_f(0)$ in the other case and this contradicts (15).

This implies that the family $\{T[f_n]\}$ is equicontinuous. By Arzela-Ascoli theorem it follows that

$$\lim_{n \rightarrow \infty} \|T[f_n] - T[f]\| = 0.$$

Thus $T[f]$ is continuous and for $\epsilon = \min_t T[f](t)$, there is n_0 such that $\|T[f_n] - T[f]\| \leq \epsilon/2$ and therefore $T[f_n](t) \geq \epsilon/2$ for $n \geq n_0$ and $t \in [0, 2\pi]$.

Moreover, since f_n is a diffeomorphism, for n sufficiently large there holds the following inequality

$$J_{w_n}(e^{i\tau}) = f'_n(\tau)T[f_n](\tau) > 0, \tau \in [0, 2\pi].$$

Since $f_n \in C^\infty$, it follows that $w_n = P[F_n] \in C^1(\overline{\mathbf{U}})$. Therefore all the conditions of Proposition 1.1 are satisfied. This means that w_n is a harmonic diffeomorphism of the unit disk onto the domain D .

Since, by a result of Hengartner and Schober [6], the limit function w of a locally uniformly convergent sequence of univalent harmonic mappings w_n on \mathbf{U} is either univalent on \mathbf{U} , is a constant, or its image lies on a straight-line, we obtain that $w = P[F]$ is univalent, because F is a surjective function of \mathbf{T} onto γ . The proof is completed. \square

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